MATH 2060A **Solution to Quiz 4a** **Spring 2019**

Name: ______________________

1. (10 marks) Show that the series $\sum_{n=1}^{\infty} x^n \sin(n\pi x)$ is uniformly convergent on $[-a, a]$ for each $a \in (0, 1)$.

   Solution. For $x \in [-a, a]$,
$$|x^n \sin(n\pi x)| \leq a^n \ .$$
   As $\sum a^n$ is convergent when $a \in (0, 1)$, by the M-Test we conclude that this series is uniformly convergent on $[-a, a]$.

   Remark. Be careful, we cannot conclude here that this series is uniformly convergent on $(-1, 1)$.

2. (5 marks) Is it continuous on $(-1, 1)$?

   Solution. From (a) we know that this series converges uniformly on $[-a, a]$ for all $a \in (0, 1)$ and as $\sum_{k=1}^{n} x^k \sin(k\pi x)$ is continuous on $[-a, a]$ for all n, we conclude from Theorem 3.6' or Continuity Theorem that $\sum_{n=1}^{\infty} x^n \sin(n\pi x)$ is continuous on $[-a, a]$ for all $a \in (0, 1)$. Therefore, it is also continuous on $(-1, 1)$. (Every point $x \in (-1, 1)$ is contained in $[-a, a]$ for some $a \in (0, 1)$.)

3. (5 marks) Is it differentiable on $(-1, 1)$?

   Solution. Let $s_n(x)$ be the n-th partial sum of the series in (a). Then
$$s_n'(x) = \sum_{k=1}^{n} \left(kx^{k-1} \sin(k\pi x) + k\pi x^k \cos(k\pi x) \right) \ .$$
   For $x \in [-a, a]$,
$$\begin{aligned} \left| kx^{k-1} \sin(k\pi x) + k\pi x^k \cos(k\pi x) \right| &\leq k\pi \left(|x|^{k-1} + |x|^k \right) \\ &\leq k\pi \left(a^{k-1} + a^k \right) \\ &\leq 2\pi k a^{k-1} \ . \end{aligned}$$
   As $\sum_{k=1}^{\infty} 2k\pi a^{k-1}$ is convergent, by M-Test, the series whose partial sums are given by s_n' converges uniformly on $[-a, a]$. By Theorem 3.8' or Differentiation Theorem, $\sum_{n=1}^{\infty} x^n \sin(n\pi x)$ is differentiable on $[-a, a]$ for all $a \in (0, 1)$, and so on $(-1, 1)$.

   Remark 1. You may use Continuity Theorem and Differentiation Theorem to name Theorem 3.6' and Theorem 3.8' respectively.

   Remark 2. The convergence of $\sum ka^{k-1}$ $(a \in (0, 1))$ follows from Ratio Test or Root Test. You don't have to write down the details since the main point of this problem is not here.